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Andrew J. Callegari and Edward L. Reiss

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# Abstract

Existence and uniqueness theorems are proved for two boundary value problems for the axisymmetric deformation of a circular membrane subjected to normal pressure. The nonlinear Föppl membrane theory is employed. The shooting method is used to establish these results.
It is also shown that if the edge is free to move in the plane of the membrane then a necessary and sufficient condition for the existence of a unique solution is that the pressure is self-equilibrating. The shooting method is applied to obtain a numerical solution of the fixed edge membrane with uniform normal pressure.

## 1. Introduction.

We consider two boundary value problems for the axisymmetric deformations of circular membranes due to an axisymmetric pressure applied normal to the plane of the membrane. We employ the Föppl [1] nonlinear membrane theory. The first problem, which we call problem H, is for the fixed edge. This is the problem considered by Hencky [2] for uniform pressure. In the second problem (Problem S), the edge is restrained from deforming normal to the plane of the membrane and is subject to a uniform radial tension, $S > 0$, applied normal to the edge. We prove that there exists a unique solution of each of these problems. Furthermore we show that the radial stress is a monotone decreasing function. If $S = 0$, then we have Problem 0. We prove that Problem 0 has a unique solution if and only if the pressure is self-equilibrating.

The "shooting method," is used to establish these results. The method is constructive and it is the basis of a well known procedure for numerically solving two point boundary value problems. It has been used previously to prove a corresponding result for a class of second order boundary value problems [3]. However the differential equation for the circular membrane is singular, and hence Keller's theorem [3], does not apply. In addition the boundary conditions for Problem H are not considered in [3]. Our analysis, which differs from that in [3], is clearly applicable to a class of singular, two point boundary value problems.

Previous existence and uniqueness theorems for Problems H and S, using an iterative procedure due to Chaplygin, have been given in [4] and [5]. It is suggested in [6] that corresponding results have been obtained for Problem O. Dicky's analysis [7], which is related to that in a previous study of circular plates [8], is for Problems H and S with uniform pressure. Furthermore, for Problem S it requires that $S > S_0 > 0$, where $S_0$ is a fixed number depending on the pressure. A simple uniqueness proof using elementary arguments is sketched in [9].

In Section 2, we formulate the boundary value problems. Since the differential equation for the circular membrane is singular, we cannot employ the standard existence and uniqueness theorems [10] for initial value problems. Thus in Section 3 we establish basic theorems for the initial value problem. The main results for the boundary value problems then follow easily by the shooting method. They are proved in Section 4. In Section 5 we apply the shooting method to obtain a numerical solution of Problem H with uniform pressure.

## 2. Formulation.

A circular membrane of radius R and thickness t is subjected to a normal pressure $p(r)$, where r is the radial coordinate. We assume that the deformation is axisymmetric. This implies that the radial and circumferential membrane stresses $\sigma_r$ and $\sigma_\theta$ and the radial and normal displacements U and W are functions of r only and that the circumferential displacement and the shear stress vanish. Then Föppl's membrane equations [1] can be reduced to the single nonlinear ordinary differential equation for the dimensionless radial stress s,

$$(1) \qquad Ls \equiv (x^3 s')' + Q/s^2 = 0 \, .$$

Here we have introduced the following notation

$$(2) \qquad \begin{aligned} &x \equiv \frac{r}{R}\,, \quad 0 \leq x \leq 1\,, \qquad s(x) \equiv \sigma_r(r)/E \\ &Q(x) \equiv (Q_0/x)\Big[\int_0^{Rx} (p(\xi)/E)\xi \, d\xi\Big]^2 \geq 0, \quad Q_0 \equiv (2t^2R^2)^{-1}, \end{aligned}$$

where E is Young's modulus. We assume that $p(r)$ is piecewise continuous on $[0,1]$. Thus, $Q(x)$ has a piecewise continuous derivative and from (2) there is a positive constant H such that

$$(3) \qquad Q \leq Hx^3 \,, \qquad 0 \leq x \leq 1 \, .$$

The dimensionless normal displacement v and the dimensionless circumferential stress $s_c$ are defined by

$$v(x) \equiv W(r)/(2^{1/2}R), \quad s_c(x) \equiv \sigma_\theta(r)/E .$$

They are determined from the equilibrium equations

(4) $$v' = s^{-1}(Q/x)^{1/2} , \qquad s_c = (xs)'.$$

To complete the formulation, conditions are required at the center x = 0 and the edge x = 1 of the membrane. Symmetry and boundedness of the stresses and displacements imply that s(0) is bounded and that

(5) $$s'(0) = 0 .$$

At the edge we consider two different boundary conditions:

(6a) $$s(1) = S > 0$$

(6b) $$s'(1) + as(1) = 0 , \quad a \equiv 1 - \nu > \frac{1}{2} ,$$

where $\nu$ is Poisson's ratio. We also require that in both cases v(1) = 0. This is the initial condition to determine v from (4). The condition (6b) implies that the radial displacement vanishes at the edge. This is the fixed edge case studied by Hencky [2]. The prescribed constant S is proportional to the radial stress applied to the edge.

Thus we define two boundary value problems. They consist of determining functions $s(x)$ with continuous first derivatives on the closed unit interval $[0,1]$ which satisfy (1) on $(0,1)$ and (5) and, for Problem S condition (6a), and for Problem H condition (6b). Since $s$ satisfies (1) on $(0,1)$ and $Q$ is continuous, $s''$ is continuous on $(0,1)$. The requirement that $s'(x)$ is continuous on $[0,1]$ insures the boundedness and continuity of $v$ and $s_c$ on $[0,1]$.

The existence of a unique solution of these boundary value problems will now be demonstrated by using the "shooting method." Thus we consider the initial value problem: Determine a function $w(x;c)$ with continuous second derivatives which satisfies

$$(7) \qquad \begin{aligned} Lw &= 0 , & x &> 0 , \\ w(0;c) &= c , & w'(0;c) &= 0 . \end{aligned}$$

The basic idea of the shooting method is to show that there exists a unique value $c_0$ of $c$ such that

$$(8) \qquad w(1;c_0) = S , \quad \text{for Problem S} ,$$

and a unique value $c_0$ such that

$$(9) \qquad w'(1;c_0) + aw(1;c_0) = 0 , \quad \text{for Problem H} .$$

Then $s(x) = w(x;c_0)$ will be the unique solution of the boundary value problem.

To apply the shooting method we require basic properties of the solutions of the initial value problem (7). Since the differential equation is singular, we cannot apply standard theorems [10]. In the next section we shall therefore analyze (7).

## 3. The Initial Value Problem.

The function $Q(x)$ is defined in (2.2) for $x$ in the unit interval. It is convenient to extend $Q$ continuously to $x \geq 1$ by the definition

$$Q(x) \equiv (Q_0/x)\left[\int_0^R \xi p \, d\xi + x^2 - 1\right]^2, \quad x \geq 1 . \tag{1}$$

Then the inequality (2.3) is preserved for all positive $x$, i.e.

$$Q(x) \leq Hx^3 , \qquad x \geq 0 . \tag{2}$$

It can be shown by direct calculation that (2.7) is equivalent to the integral equation

$$w(x;c) = c - \int_0^x k(x,\xi) w^{-2} \, d\xi , \tag{3a}$$

$$k(x,\xi) \equiv \tfrac{1}{2} Q(\xi)(\xi^{-2} - x^{-2}) \geq 0 , \quad \text{for } 0 \leq \xi < x , \tag{3b}$$

in the sense that every solution of (2.7) is a solution of (3) and conversely. By differentiating (3a) we obtain

$$w'(x;c) = - x^{-3} \int_0^x Q w^{-2} \, d\xi . \tag{4}$$

Thus, every solution of (3) is a monotone decreasing function. If $Q \equiv 0$ in $[0,1]$, then from (3), $w = c$ is the unique solution of (2.7). Then Problem S has the unique solution $s(x) = w(x;S) = S$, and Problem H has the unique solution $s(x) = w(x;0) = 0$. We shall hereafter assume that $Q(x) \not\equiv 0$ in $[0,1]$.

The basic properties of the initial value problem (2.7) are contained in the Theorems 1-5.

THEOREM 1. *For every* $S > 0$ *and every* $c > S$, *there is a unique number* $x_0 > 0$ *such that only one solution of* (2.7) *exists in the interval* $0 \leq x \leq x_0$ *and* $w(x_0;c) = S$. *The solution is a positive and monotone decreasing function of* $x$ *in this interval.*[*]

Proof: We first demonstrate the uniqueness in any interval in which $w$ does not vanish. For a fixed $c$, let $w_1$ and $w_2$ be two distinct non-vanishing solutions in the interval $0 \leq x \leq \alpha$, where $\alpha > 0$. Since $c > 0$ it is always possible to determine such an interval by making $\alpha$ sufficiently small. Then $D(x) \equiv w_1 - w_2$ satisfies the linear integral equation:

$$(5) \qquad D(x) = \int_0^x k(x;\xi) \left[ \frac{w_1(\xi) + w_2(\xi)}{w_1^2(\xi) \; w_2^2(\xi)} \right] D(\xi)\, d\xi \,, \quad 0 \leq x \leq \alpha .$$

Since the kernel is continuous in the triangle $0 \leq x \leq \alpha$, $0 \leq \xi \leq x$, the Volterra equation (5) has a unique solution. Thus $D \equiv 0$ in $0 \leq x \leq \alpha$ and the initial value problem has no more than one solution in this interval.

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[*] We need only consider $c > 0$ since for $c \leq 0$ a solution of (2.7) cannot solve the boundary value problems.

To prove the existence* we consider a strictly monotone decreasing sequence of initial points $\{a_n\}$ with $a_1 < 1$ and $\lim_{n \to \infty} \{a_n\} = 0$. For each n, the corresponding initial value problem for $w_n$

$$(6) \qquad Lw_n = 0, \quad w_n(a_n;c) = c, \quad w_n'(a_n;c) = 0 \;, \quad n = 1,2,\ldots \;,$$

has, by standard theorems [10], a unique solution in an interval $0 < a_n < x \leq \alpha_n$ in which $w_n$ does not vanish. Clearly, (6) and the integral equation

$$(7) \qquad w_n(x;c) = c - \int_{a_n}^{x} k(x,\xi) w_n^{-2}(\xi)\, d\xi \;, \qquad a_n \leq x \leq \alpha_n$$

are equivalent. In addition we have by differentiating (7)

$$(8) \qquad w_n' = - x^{-3} \int_{a_n}^{x} Q w_n^{-2}\, d\xi \;.$$

If $Q(a_n) \neq 0$ then from (8), $w_n$ is a strictly monotone decreasing function of x. Suppose that $Q(x) \equiv 0$ for x in the interval $a_n \le x \le X_n$ and $Q(x) \not\equiv 0$ in any interval with $X_n$ as the left end point. Then $w_n = c$ in $a_n \le x \le X_n$ and it is strictly monotone decreasing when $x > X_n$. For every positive number $S < c$ either $w_n(x;c) = S$ for a unique $x = x_n$ or else $\lim\limits_{x \to \infty} w_n' = 0$ with $\lim\limits_{x \to \infty} w_n = A \ge S$. The latter possibility cannot occur since by (8) $\lim\limits_{x \to \infty} w_n' = -\left(\frac{1}{3A^2}\right) \lim\limits_{x \to \infty} \frac{Q}{x^2}$ and using (1) $\lim\limits_{x \to \infty} w_n' = -\infty$. Thus in (6) and (7), $\alpha_n > x_n$.

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* An existence proof can also be obtained by showing that the initial value problem has a convergent power series expansion in a sufficiently small neighborhood of $x = 0$. Then choosing an initial point $x = x^* \neq 0$, in this interval, the coefficients in the differential equation are not singular at $x^*$ and standard theorems [10] can be applied for $x > x^*$.

LEMMA 1. *The sequence* $\{x_n\}$ *is monotone decreasing and converges to* $x_0 > 0$.

Proof: Suppose the sequence is not monotone decreasing. Then for some n, $x_{n+1} > x_n$ and there is a point $x^* < x_n$ at which $w_n(x^*; c) = w_{n+1}(x^*;c)$. This implies using (7) that

$$\int_{a_{n+1}}^{a_n} k(x^*,\xi) w_{n+1}^{-2} \, d\xi = \int_{a_n}^{x^*} k(x^*,\xi)(w_n^{-2} - w_{n+1}^{-2}) \, d\xi \le 0$$

since $w_{n+1} \le w_n$ in $a_n \le x \le x^*$. If $Q \not\equiv 0$ in $(a_n, a_{n+1})$ then this is a contradiction because $a_{n+1} < a_n$. If $Q \equiv 0$ in $(a_n, a_{n+1})$ then $w_{n+1} \equiv w_n$ in $[a_n, x_n]$ and $x_{n+1} = x_n$. Therefore $x_{n+1} \le x_n$ and $\{x_n\}$ is monotone decreasing. Since $x_n > 0$ for all n, the sequence converges to $x_0$. The limit $x_0$ must be positive. This follows by using (7) in $w_n(x_n;c) = S$ to obtain

$$(9)\quad 0 < c - S = \int_{a_n}^{x_n} k(x_n,\xi)w_n^{-2}\, d\xi \le \frac{H}{2S^2}\int_{a_n}^{x_n} \xi^3(\xi^{-2}-x_n^{-2})\, d\xi$$

$$\le \frac{H}{8S^2}(x_n^2 - a_n^2) .$$

We have used $w_n \geq S$ and (2) in estimating in (9). If $x_0 = 0$ then the right side can be made arbitrarily small by making n sufficiently large since $a_n \longrightarrow 0$. This is impossible because the left side is positive and the lemma is established.

It is convenient to introduce the infinite sequence of functions $\{t_n(x)\}$ defined on the fundamental interval $0 \leq x \leq x_0$ for $n = 1,2,\ldots,$ by

$$(10)\qquad t_n(x) = \begin{cases} c, & 0 \leq x \leq a_n ; \\ w_n(x;c), & a_n \leq x \leq x_0 \quad \text{if } a_n < x_0 . \end{cases}$$

The sequence is uniformly bounded by c since $w_n$ is monotone decreasing. If the sequence is uniformly Lipschitz continuous* with respect to x and n on $[0,x_0]$, then by Arzela's theorem there exists an infinite subsequence $\{t_{n_j}(x)\}$ which converges uniformly to a continuous function $w_0(x,c)$ on $[0,x_0]$. The function $w_0$ is a solution $w(x;c)$ of (2.7). To see this let $n \rightarrow \infty$ over the subsequence. Since the subsequence converges uniformly and the integrand in (7) is a continuous function of $w_{n_j}$ we can interchange the limit and integral. Hence, since $a_n \to 0$, $w_0$ satisfies (3a) and is a solution of the initial value problem* (2.7) for $0 \leq x \leq x_0$. Furthermore, $w_0(x_0;c) = S$ and the monotonicity is a consequence of (4).

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* The sequence is uniformly Lipschitz continuous with respect to x and n if the Lipschitz constant K is independent of x and n.

Therefore to prove the theorem we need only establish that the sequence (10) is uniformly Lipschitz continuous in x and n. This is easily done by observing from (8) that $w_n'$ is continuous and bounded on $[a_n, x_0]$. For any x in this interval we have

$$(11) \quad |w_n'| \leq \frac{1}{x^3S^2}\int_{a_n}^{x} Q^2 \, d\xi \leq \frac{1}{x^3S^2}\int_{0}^{x} Q^2 \, d\xi \leq \frac{H}{S^2}\frac{x}{4} \leq \frac{Hx_0}{4S^2} = K ,$$

where we have used (2) and the inequality $w_n \geq S$. Since $w_n' = 0$ for $0 \leq x \leq a_n$, the bound (11) holds for all x in the fundamental interval. Since K is independent of x and n it is the required Lipschitz constant.

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*Every convergent subsequence converges to a solution of (2.7) in $0 \leq x \leq x_0$. Since (2.7) has a unique solution, the entire sequence (10) converges uniformly to the same limit.

THEOREM 2. *Let* $w(x;c)$ *be the solution of the initial value problem* (2.7). *Then for every* $S > 0$ *and every* $c > S$, $w(x;c)$ *and* $w'(x;c)$ *are continuous functions of* $c$ *for every* $x$ *in the fundamental interval* $[0,x_0(c)]$.

Proof: Consider any two values of $c > S$, $c_1$ and $c_2$ and let $w_1(x) \equiv w(x;c_1)$ and $w_2(x) \equiv w(x;c_2)$. Then using (3), the difference $v(x) \equiv w_1 - w_2$ satisfies the integral equation

$$(12) \quad v(x) = c_1 - c_2 - \int_0^x k(x;\xi) \left| \frac{w_1(\xi) + w_2(\xi)}{w_1^2(\xi) w_2^2(\xi)} \right| v(\xi)\, d\xi ,$$

for $0 \leq x \leq m \equiv \min\,[x_0(c_1), x_0(c_2)]$. Taking absolute values in (12) and using (2) and the inequalities $w_i \leq c_i$, $w_i \geq S$, $i = 1,2$, to estimate the integrand we obtain

$$(13) \quad |v| \leq |c_1 - c_2| + 2A \int_0^x \xi |v|\, d\xi, \qquad 2A \equiv H(c_1 + c_2)/(2S^4) .$$

Applying Gronwall's inequality [10] to (13) yields

$$(14) \qquad |v| \leq |c_1 - c_2|\, e^{Ax^2} .$$

The continuity of $w$ as a function of $c$ for fixed $x$ is a consequence of (14). The continuity of $w'$ follows directly from the integral relation (4) and the continuity of $w$.

The final three theorems are concerned with the variation of the point $x_0(c)$. We show that $x_0(c)$ is continuous and strictly monotone increasing for $c \geq S$ and $x_0(\infty) = \infty$. Clearly if $Q(0) \neq 0$, $\lim_{c \to S} x_0(c) = 0$. If $Q(x) \equiv 0$ for $x$ in the interval $0 \leq x \leq X < 1$ and $Q(x) \not\equiv 0$ for any interval with $X$ as the left end point, then $\lim_{c \to S} x_0(c) = X < 1$.

THEOREM 3. $x_0(c)$ is strictly monotone increasing for $c \geq S$.

Proof: Assume the contrary, i.e. there are two values $c_1$ and $c_2$ of $c$ for which $c_2 > c_1 \geq S$ and $x_0(c_1) \geq x_0(c_2)$. Then there must be a value $x^*$ in the interval $0 < x \leq x_0(c_2)$ such that

$$(15a) \qquad w(x^*;c_1) = w(x^*;c_2) ,$$

$$(15b) \qquad w(x;c_1) < w(x;c_2) , \qquad 0 \leq x < x^*.$$

Substituting the integral equation (3a) in (15a) we obtain

$$(16) \quad 0 < c_2 - c_1 = \int_0^{x^*} k(x^*,\xi)[w^{-2}(\xi;c_2) - w^{-2}(\xi;c_1)]\, d\xi .$$

Since $k \geq 0$ it follows from (15b) that the integrand in (16) is non-positive. Hence (16) yields a contradiction and the theorem is proved.

THEOREM 4. $\lim_{c \to \infty} x_0(c) = \infty$.

Proof: From (3a) we have, $w(x_0(c),c) = S$ $= c - \int_0^{x_0} k(x_0,\xi)w^{-2}(\xi;c)\, d\xi$. Since $w(\xi;c) \geq S$ in $[0,x_0]$, we have $c - S \leq \frac{1}{S^2} \int_0^{x_0} k(x_0,\xi)\, d\xi$. Therefore when $c \to \infty$ the integral $\to \infty$. Since the integrand is bounded this implies that $x_0 \to \infty$.

THEOREM 5. $x_0(c)$ is continuous for $c \geq S$.

Proof: From Theorem 2 we have that $w$ and $w'$ are continuous functions of $c$. Since $w'(x_0(c);c) < 0$, application of the implicit function theorem to $w(x_0;c) = S$ proves the theorem.

## 4. The Boundary Value Problems.

The basic theorem for Problem S follows easily from the properties of the initial value problem established in the previous section.

THEOREM S. *Problem* S *has a unique solution* s(x). *It is a positive and monotone decreasing function.*

Proof: Since $x_0(c)$ is monotone and varies between $X < 1$ and infinity,* there is a unique value $c_0$ of c such that $x_0(c) = 1$. The positive and monotone decreasing function $s(x) \equiv w(x,c_0)$ is the unique solution of (2.7) that satisfies the boundary condition (2.6a) and hence is the required solution of the Problem S.

When $S = 0$ in (2.6a) $Q/s^2$ and hence $s'$ may be singular at $x = 1$ and the above analysis must be modified. We shall call Problem S with $S = 0$, Problem 0. We consider a sequence of boundary value problems,

$$Ls_n = 0 \ , \quad s_n'(0) = 0 \ , \quad s_n(1) = S_n > 0 \ , \quad n = 1,2,\ldots, \tag{1}$$

where $\{S_n\}$ is a monotone strictly decreasing sequence of positive numbers that converges to zero. It follows from Theorem S that for each n, (1) has a unique solution $s_n(x)$

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*Here X is the value defined in Section 3 immediately preceding Theorem 3.

which also solves the integral equation

$$(2) \qquad s_n(x) = c_n - \int_0^x k(x,\xi)\, s_n^{-2}\, d\xi \ .$$

Here $c_n$ is the uniquely determined initial value $c_n = s_n(0)$. In the Appendix we show that the sequence $\{s_n(x)\}$ converges uniformly on $[0,1]$ to the function $s(x)$ with $s(1) = 0$. The limit function is independent of the choice of the sequence $\{S_n\}$. Therefore taking limits in (2) we observe that $s(x)$ satisfies the integral equation

$$(3) \qquad s = c - \int_0^x k(x,\xi)\, s^{-2}\, d\xi \ ,$$

where $c = \lim_{n\to\infty} c_n$ and

$$(4) \qquad s' = -x^3 \int_0^x Q\, s^{-2}\, d\xi \ .$$

Since s satisfies (3), it is a solution of the following boundary value problem which we call Problem B. It consists of determining functions with continuous first derivatives on (0,1) which satisfy (2.1) on (0,1) and (2.5) and (2.6a) with $S = 0$.* As observed by Srubshchik [6] a solution of Problem B is not necessarily a solution of

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* If $p(r)$ is a homogeneous function of degree m, then the existence theorem for this boundary value problem can be easily established by considering the change of variables $y = c^{-a}x$, $G(y) = s(x)/c$ where $a \equiv 3/(2m+5)$.

Problem 0 since $s'(1)$ may be unbounded.* We now give a necessary and sufficient condition for Problem 0 to have a unique solution.

THEOREM 0. Problem 0 has a unique solution if and only if $Q(1) = 0$.

Proof: Let $s(x)$ be the unique solution of Problem 0. Then $s(x)$ is also a solution of Problem B and $s'(x)$ is continuous on $[0,1]$. Hence from (4), $-s'(1) = I \equiv \int_0^1 Q\, s^{-2}\, d\xi$ exists. Assume that $Q(1) \neq 0$. Since I exists, then so does $\int_0^1 s'\, s^{-2}\, d\xi = -\int_0^1 (s^{-1})'\, d\xi$. However, this integral is unbounded and we have a contradiction. Thus $Q(1) = 0$ is a necessary condition for Problem 0 to have a unique solution. To prove the sufficiency we integrate (4), for any solution s of Problem B, from x to 1 and use the boundary condition $s(1) = 0$ and the inequality $s(x) < c$ to obtain

$$s(x) > c^{-2} f(x) \equiv c^{-2} \int_x^1 \eta^{-3} \int_0^\eta Q\, d\xi\, d\eta \,. \tag{5}$$

The integral (5) exists since $\lim_{\eta \to 0} \eta^{-3} \int_0^\eta Q\, d\xi = 0$. Thus by inserting (5) in $s'(1)$, we see that it is bounded

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[*] We recall, see Section 2, that Problem 0 requires that $s'$ is continuous on the closed unit interval.

if $\int_0^1 Q\, f^{-2}\, d\xi$ exists. We observe from (2.2) that $Q(1) = 0$ implies that $Q'(1) = 0$. The function $f(x)$ is monotone decreasing, $f(1) = 0$ and $f'(1) \neq 0$. Hence $Q f^{-2}$ is bounded at $x = 1$ and Problem 0 has a solution. The uniqueness is established in the usual way by assuming that Problem 0 has two solutions $s_1$ and $s_2$. They both satisfy (3). Since $s_1(1) = s_2(1) = 0$, we have from (3)

$$(6) \qquad s_1(0) - s_2(0) = \int_0^1 k(1,\xi)\, [s_1^{-2} - s_2^{-2}]\, d\xi\ .$$

From Theorem 1 we may choose $s_1(0) > s_2(0)$ and by the proof of Theorem 3, $s_1(x) > s_2(x)$ for $0 \leq x < 1$. Thus (6) yields a contradiction and $s_1(x) \equiv s_2(x)$. The condition $Q(1) = 0$ implies that the pressure has a zero resultant, i.e. $p(r)$ is self-equilibrating.

The corresponding result for Problem H is established by showing that for the function g defined by

$$(7) \qquad g(x;c) \equiv x^3\, w'(x;c) + aw(x;c)\ ,$$

where w is the solution of the initial value problem (2.7), there is a unique value, $c_0$, of c such that

$$(8) \qquad g(1;c_0) = 0.$$

Then $s(x) \equiv w(x;c_0)$ is the unique solution of Problem H. We shall now obtain the necessary properties of g.

From the definition (7), g and g' are obviously continuous functions in $[0,x_0]$ for any $S > 0$. Substituting the integral relations (3.3) and (3.4) in (7) we obtain

$$(9) \qquad g = ac - \int_0^x K(x,\xi)\, w^{-2}(\xi)\, d\xi \ , \quad K(x,\xi) \equiv ak(x,\xi) + Q\,.$$

Differentiation of (9) gives

$$(10) \qquad g' = -Q/w^2 - ax^{-3} \int_0^x (Q/w^2)\, d\xi \leq 0\ .$$

The strict inequality holds in (10) if $Q \not\equiv 0$ in $(0,x)$. Thus for each c, g is a monotone decreasing function of x. Therefore if $c < 0$ it cannot have a positive zero and we need only consider $c \geq 0$.

From (3.4) and (3.1) we observe that $w' < 0$ and for sufficiently large x, w' assumes arbitrarily large negative values. Therefore by making S sufficiently small, we conclude from (7) that g has a zero at $x = y(c)$ in the interval $0 < y < x_0$. The strict inequality must hold in (10) in some neighborhood of y. Therefore $x = y$ is the unique zero of g for fixed c. The function $y(c)$ has the properties summarized in Theorem 6.

THEOREM 6. $y(c)$ is a continuous, strictly monotone increasing function for $c > 0$ and $\lim_{c \to 0} y(c) = Y < 1$, $\lim_{c \to \infty} y(c) = \infty$.

The proof of this theorem will not be presented since it employs arguments similar to those used in establishing Theorems 3, 4 and 5.

Thus from Theorem 6 we conclude that there is a unique value $c_0$ of $c$ for which $y(c_0) = 1$ and hence $g(1;c_0) = 0$. Therefore we have established

THEOREM H. Problem H has a unique solution $s(x)$. It is a positive and monotone decreasing function.

## 5. Numerical Solution.

The method of proof used in the previous sections is constructive. We shall now apply it to obtain a numerical solution of Problem H with p = constant. From (2.2),

$$Q(x) = Px^3 , \qquad P \equiv (1/8)(pR/Et)^2 .$$

We define a new dependent variable T by

$$T(x) \equiv s(x)/P^{1/3} .$$

Then Problem H is given by

$$(1) \qquad (x^3T')' = - x^3/T^2 , \qquad T'(0) = T'(1) + \mu T(1) = 0 .$$

From Theorem H in the previous section, it follows that there exists a unique value of $c = T(0)$ such that

$$(2) \qquad f(c) \equiv T'(1;c) + \mu\, T(1,c) = 0 .$$

We determine the value of c numerically by solving (2) by Newton's method. To do this we must consider the variational equations with respect to c of the initial value problem. We have solved the initial value problem and the variational problem numerically by the Runge-Kutta method.* For details of the numerical application of the shooting method see [11]. The values of $T(x)$ and $T_c(x) \equiv s_c(x)/P^{1/3}$ obtained from this computation are graphed in Figure 1. The numerical value of $c$ that we obtained is 0.8669531.

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[*] In the calculations we used the value $\nu = 0.32$. They were performed on the CDC 6600 Computer of the AEC Computing and Applied Mathematics Center of the Courant Institute of Mathematical Sciences. The computation was done by Dr. L. Bauer.

# References

1. Föppl, A.: Vorlesungen über technisch Mechanik, Vol. 5. Leipzig: Teubner, 1907.
2. Hencky, H.: Über der Spannungszustand in Kreisrunden Platten. Z. Math. Phys. 63, 311-317 (1915).
3. Keller, H. B.: Existence Theory for Two Point Boundary Value Problems. Bull. Amer. Math. Soc. 72, 728-731 (1966).
4. Srubshchik, L. S. and Iudovich, V. I.: Asymptotic Representation of Large Deflections of a Circular Symmetrically Loaded Plate. Sib. Mat. Zh. 4, 657-672 (1963).
5. Srubshchik, L. S. and Iudovich, V. I.: The Asymptotic Integration of the System of Equations for the Large Deflection of Symmetrically Loaded Shells of Revolution. Prikl. Mat. Mekh. 26, 913-922 (1962). Translation in J. Appl. Math. and Mech. 26, 1378-1392 (1962).
6. Srubshchik, L. S.: On the Existence of a Solution to the Membrane Problem of the Equilibrium of a Circular Membrane. Prikl. Mat. Mekh. 30, 576-579 (1966). Translation in J. Appl. Math. and Mech. 30, 685-689 (1966).
7. Dickey, R. W.: The Plane Circular Elastic Surface under Normal Pressure. Arch. Rat. Mechs. Analys. 26, 219-236 (1967).
8. Keller, H. B. and Reiss, E. L.: Iterative Solutions for the Non-Linear Bending of Circular Plates. Comm. on Pure and Appl. Math. 11, 273-292 (1958).
9. Reiss, E. L.: A Uniqueness Theorem for the Nonlinear Axisymmetric Bending of Circular Plates, AIAA Jour. 1, 2650-2652 (1963).
10. Coddington, E. A. and Levinson, N.: Theory of Ordinary Differential Equations. New York: McGraw-Hill, 1955.
11. Keller, H. B.: Numerical Methods for Two-Point Boundary-Value Problems. Waltham, Mass.: Blaisdell (in press).

Appendix.

THEOREM. The sequence of functions $\{s_n(x)\}$ defined by (4.1) is uniformly convergent on [0,1].

Proof: To prove the theorem we must first show that

$$s_{n+1}(x) < s_n(x) \quad \text{for} \quad 0 \leq x \leq 1 \;, \quad n = 1,2,\ldots \; . \tag{1}$$

First we consider $x = 0$ where $s_n(0) = c_n$ and we assume that[†] $c_{n+1} > c_n$. Then there is a point $x^*$ in (0,1) at which

$$s_{n+1}(x^*) = s_n(x^*) \;, \tag{2}$$

and $s_{n+1} > s_n$ for $0 \leq x < x^*$. Inserting (4.2) in (2), we obtain

$$c_{n+1} - c_n = \int_0^{x^*} k(x,\xi)\,(s_{n+1}^{-2} - s_n^{-2})\, d\xi \; . \tag{3}$$

The right hand side of (3) is negative. This provides a contradiction and hence $c_{n+1} < c_n$. Since $c_n > 0$, the sequence $\{c_n\}$ converges. By a similar argument we can show that $s_{n+1} \not> s_n$ for any x in (0,1). Therefore (1) is established. We now consider for $n > m$ the quantity

---

[†] We need not consider $c_n = c_{n+1}$ because of Theorem 1 and $s_{n+1} < s_n$.

$(s_m - s_n)'$. Using (4.2), it is given by

$$(4) \qquad (s_m - s_n)' = s_m' - s_n' = -\frac{1}{x^3}\int_0^x Q(s_m^2 - s_n^2)\, d\xi \,.$$

Since (1) is true for all n, the right hand side of (4) is positive and $s_m - s_n$ is monotone increasing. Therefore we have

$$(5) \qquad s_m(x) - s_n(x) < S_m - S_n \quad \text{for} \quad 0 \leq x \leq 1 \,.$$

Since the sequence $\{S_n\}$ converges to zero the sequence $\{s_n(x)\}$ is uniformly convergent and the theorem is proved.

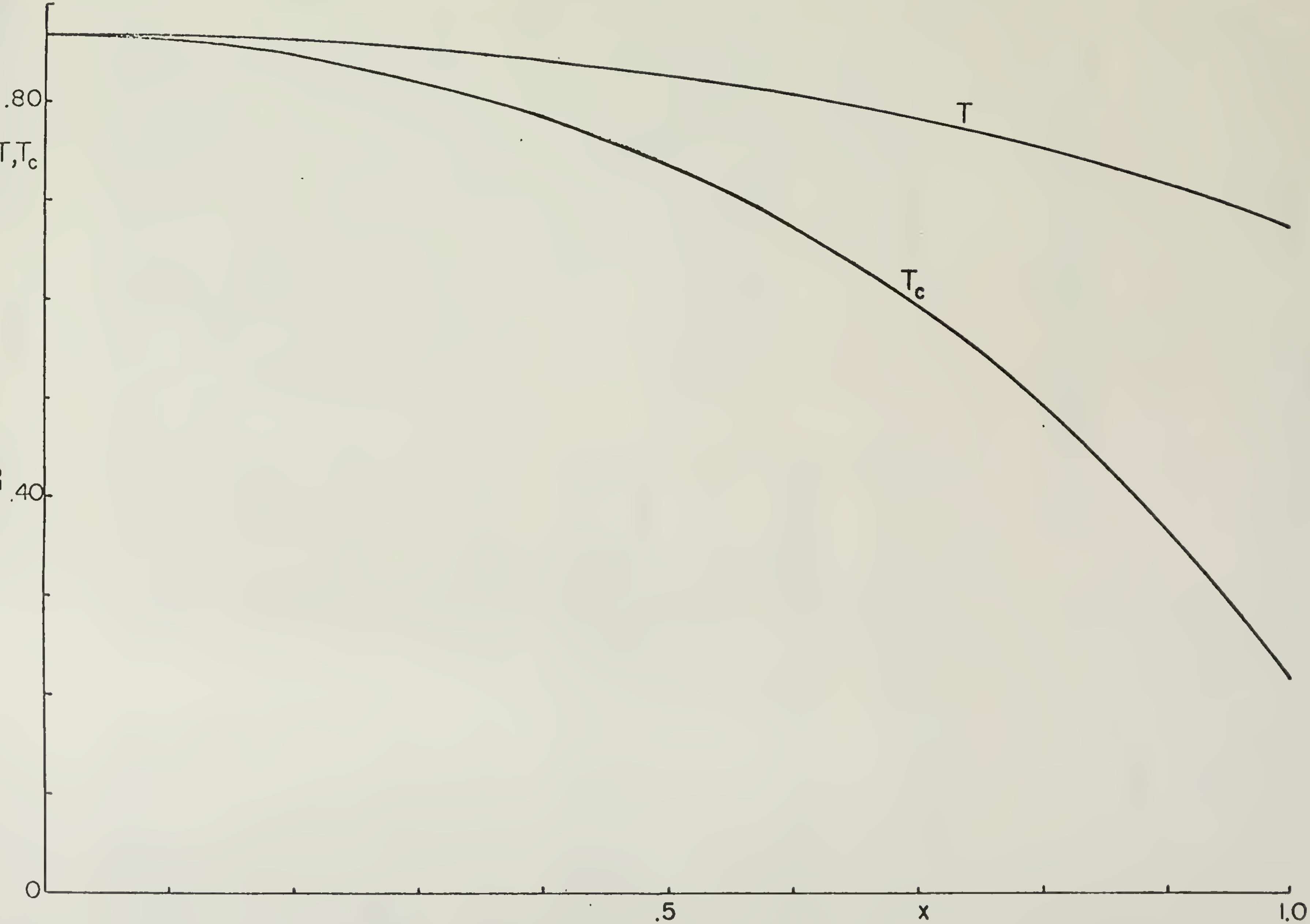

Fig. 1. Variation of the dimensionless radial stress T and the dimensionless circumferential stress $T_c$ for Problem H with constant pressure and $\nu$ = .32

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Existence and uniqueness theorems are proved for two boundary value problems for the axisymmetric deformation of a circular membrane subjected to normal pressure. The nonlinear Föppl membrane theory is employed. The shooting method is used to establish these results. It is also shown that if the edge is free to move in the plane of the membrane then a necessary and sufficient condition for the existence of a unique solution is that the pressure is self-equilibrating. The shooting method is applied to obtain a numerical solution of the fixed edge membrane with uniform normal pressure.

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Mr. K. W. Bills, Jr.
Dept. 4722, Bldg. 0525
Aerojet-General Corporation
P. O. Box 1947
Sacramento, California 95809

Dr. James H. Wiegand
Senior Dept. 4720, Bldg. 0525
Ballistics & Mech. Properties Lab.
Aerojet-General Corporation
P. O. Box 1947
Sacramento, California 95809

Dr. John Zickel
Dept. 4650, Bldg. 0227
Aerojet-General Corporation
P. O. Box 1947
Sacramento, California 95809

Mr. J. S. Wise
Aerospace Corporation
P. O. Box 1308
San Bernardino, California 92402

Dr. Vito Salerno
Applied Technology Assoc., Inc.
29 Church Street
Ramsey, New Jersey 07446

Library Services Department
Report Section, Bldg. 14-14
Argonne National Laboratory
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Dr. E. M. Kerwin
Bolt, Beranek, & Newman, Inc.
50 Moulton Street
Cambridge, Massachusetts 02138

Dr. M. C. Junger
Cambridge Acoustical Associates
129 Mount Auburn Street
Cambridge, Massachusetts 02138

Dr. F. R. Schwarzl
Central Laboratory T.N.O.
134 Julianalaan
Delft, Holland

Industry & Research Inst. (cont'd.)

Mr. Ronald D. Brown
Applied Physics Laboratory
Chemical Propulsion Agency
8621 Georgia Avenue
Silver Spring, Maryland 20910

Research and Development
Electric Boat Division
General Dynamics Corporation
Groton, Connecticut 06340

Supervisor of Shipbuilding, USN,
and Naval Insp. of Ordnance
Electric Boat Division
General Dynamics Corporation
Groton, Connecticut 06340

Dr. L. H. Chen
Basic Engineering
Electric Boat Division
General Dynamics Corporation
Groton, Connecticut 06340

Mr. Ross H. Petty
Technical Librarian
Allegany Ballistics Lab.
Hercules Powder Company
P. O. Box 210
Cumberland, Maryland 21501

Dr. J. H. Thacher
Allegany Ballistic Laboratory
Hercules Powder Company
Cumberland, Maryland 21501

Dr. Joshua E. Greenspon
J. G. Engr. Research Associates
3831 Menlo Drive
Baltimore, Maryland 21215

Mr. R. F. Landel
Jet Propulsion Laboratory
4800 Oak Grove Drive
Pasadena, California 91103

Mr. G. Lewis
Jet Propulsion Laboratory
4800 Oak Grove Drive
Pasadena, California 91103

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Dry Dock Company
Newport News, Virginia 23607

Mr. E. A. Alexander(Research Dept.(991)
Rocketdyne Division
North American Aviation, Inc.
6633 Canoga Avenue
Canoga Park, California 91304

Mr. Cezar P. Nuguid
Deputy Commissioner
Philippine Atomic Energy Commission
Manila, Philippines

Mr. S. C. Britton
Solid Rocket Division
Rocketdyne
P. O. Box 548
McGregor, Texas 76657

Dr. A. J. Ignatowski
Redstone Arsenal Research Div.
Rohm & Haas Company
Huntsville, Alabama 35807

Dr. M. L. Merritt
Division 5412
Sandia Corporation
Sandia Base
Albuquerque, New Mexico 87115

Director
Ship Research Institute
Ministry of Transportation
700, SHINKAWA
Mitaka
Tokyo, JAPAN

Dr. H. N. Abramson
Southwest Research Institute
8500 Culebra Road
San Antonio, Texas 78206

Industry & Research Inst. (cont'd.)

Dr. R. C. DeHart
Southwest Research Institute
8500 Culebra Road
San Antonio, Texas 78206

Dr. Thor Smith
Stanford Research Institute
Menlo Park, California 94025

Dr. M. L. Baron
Paul Weidlinger, Consulting Engr.
777 Third Ave. - 22nd Floor
New York, New York 10017

Dr. T. C. Fan
The Rand Corporation
1700 Main Street
Santa Monica, California 90406

Dr. G. R. Makepeace
Director, Research & Engineering
Lockheed Propulsion Company
P. O. Box 111
Redlands, California 92374

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N.Y.U. Courant Institute of
Mathematical Sciences
251 Mercer St.
New York, N. Y. 10012